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Oscillations of Some Delay Differential Equations with Periodic Coefficients

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1. INTRODUCTION AND STATEMENT OF THE THEOREMS

The oscillation theory of delay differential equations has been extensively developed during the past few years (see, for example, [1–12] and the references cited therein). A delay differential equation provides a mathematical model for physical systems in which the rate of change of the systems depends not only on their present state, but also in their past history. In this paper we obtain new oscillation results for delay differential equations with periodic coefficients.

Consider the delay differential equation

$$x'(t)+a(t)[px(t-\tau)+qx(t-\sigma)]=0, \tag{E}$$

where *a is a nonnegative continuous function on the interval* $[0, \infty)$ *which is not identically zero on this interval, p and q are real numbers, and τ and σ are nonnegative constants. It will be supposed that the function a is periodic with a period $\omega>0$ and that there exist nonnegative integers μ and ν such that*

$$\tau=\mu\omega \quad \text{and} \quad \sigma=\nu\omega.$$

Consider also the delay equation

$$x'(t)+f(t)\,x(t-\tau)-g(t)\,x(t-\sigma)=0, \tag{$\tilde{\text{E}}$}$$

where *f and g are nonnegative continuous functions on* $[0, \infty)$ *which are not identically zero and ω-periodic, and τ and σ are exactly as above.*

Let $t_0 \geqslant 0$. By a *solution on* $[t_0, \infty)$ of the differential equation (E) (respectively, of the equation ($\tilde{\text{E}}$)) we mean a continuous real-valued function x defined on the interval $[t_0-\rho, \infty)$, where $\rho=\max\{\tau, \sigma\}$, which is

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differentiable on $[t_0, \infty)$ and satisfies (E) (resp. ($\tilde{\mathrm{E}}$)) for all $t \geqslant t_0$. As usual, a solution of (E) or ($\tilde{\mathrm{E}}$) is said to be *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* otherwise.

Throughout the paper, we will use the notation

$$A = \frac{1}{\omega} \int_0^{\omega} a(t)\, dt$$

and

$$F = \frac{1}{\omega} \int_0^{\omega} f(t)\, dt \qquad \text{and} \qquad G = \frac{1}{\omega} \int_0^{\omega} g(t)\, dt$$

(clearly, A, F, and G are positive constants). Moreover, with the differential equation (E) we associate the equation (which is in a sense the *characteristic equation* of (E))

$$\lambda + A(pe^{-\lambda\tau} + qe^{-\lambda\sigma}) = 0. \tag{$*$}$$

In this paper, we give a necessary and sufficient condition for the oscillation of all solutions of (E) via the equation ($*$). Also, we obtain conditions under which every nonoscillatory solution of ($\tilde{\mathrm{E}}$) tends to zero as $t \to \infty$. Moreover, we give sufficient conditions for all solutions of ($\tilde{\mathrm{E}}$) to be oscillatory. More precisely, in this paper the following results are proved.

THEOREM 1. *The following statements are equivalent:*

(I) *All solutions of* (E) *are oscillatory.*

(II) *Equation* ($*$) *has no real roots.*

THEOREM 2. *Let $c > 1$ be a constant such that $f \geqslant cg$ and assume that*

$$0 \leqslant (\tau - \sigma)G \leqslant 1. \tag{C_1}$$

Then every nonoscillatory solution of ($\tilde{\mathrm{E}}$) *tends to zero at ∞.*

THEOREM 3. *Let $c > 1$ be a constant such that $f \geqslant cg$ and assume that condition* (C_1) *holds. Moreover, assume that*

$$\tau(F - G)e > 1. \tag{C_2}$$

Then all solutions of ($\tilde{\mathrm{E}}$) *are oscillatory.*

The importance of Theorem 1 is that the coefficients of (E) are not restricted to be nonnegative. See [11] for the case of delay differential equations with nonnegative periodic coefficients.

In the special case where $a(t)=1$ for $t \geqslant 0$, Theorem 1 leads to the main result in [3].

For delay differential equations with constant nonnegative coefficients, a necessary and sufficient condition for the oscillation of all solutions is established in [12] (see also [1, 4, 8]).

When the coefficients f and g of the differential equation ($\tilde{\mathrm{E}}$) are constant, Theorems 2 and 3 lead to known results [2] (cf. also [7]).

2. Some Useful Lemmas for the Proof of Theorem 1

In this section we establish some lemmas which will be used in the proof of Theorem 1.

Lemma 1. *Assume that*

$$\tau > \sigma > 0. \tag{1}$$

Then the inequalities

$$p+q>0, \tag{2}$$

$$p>0, \tag{3}$$

and

$$m=\min_{\lambda \in \mathbb{R}} [\lambda + A(pe^{-\lambda\tau}+qe^{-\lambda\sigma})]>0 \tag{4}$$

are necessary conditions for $(*)$ *to have no real roots.*

Proof. See [3, Lemma 1].

Lemma 2. *Let* x *be a solution on an interval* $[t_0, \infty)$, $t_0 \geqslant 0$, *of the differential equation* (E) *and set*

$$z(t)=x(t)-p\int_{t-\tau}^{t-\sigma} a(s)\,x(s)\,ds \text{ for } t \geqslant \max\{t_0, \tau, \sigma\}. \tag{5}$$

Then z *is a solution of* (E) *on the interval* $[T_0, \infty)$, *where* $T_0 = \max\{t_0, \tau, \sigma\}+\max\{\tau, \sigma\}$.

Proof. Since the function a is ω-periodic and because of the fact that $\tau=\mu\omega$ and $\sigma=\nu\omega$, from (5) and (E) we obtain for $t \geqslant \max\{t_0, \tau, \sigma\}$

$$\begin{aligned} z'(t) &= x'(t)-p[a(t-\sigma)\,x(t-\sigma)-a(t-\tau)\,x(t-\tau)] \\ &= -a(t)[px(t-\tau)+qx(t-\sigma)]-pa(t)[x(t-\sigma)-x(t-\tau)] \end{aligned}$$

and consequently

$$z'(t) = -a(t)(p+q)\,x(t-\sigma) \qquad \text{for every} \quad t \geqslant \max\{t_0, \tau, \sigma\}. \tag{6}$$

On the other hand, by using again the fact that a is ω-periodic and that $\tau = \mu\omega$ and $\sigma = \nu\omega$, from (5) and (E) we derive for $t \geqslant T_0$

$$\begin{aligned}
&pz(t-\tau) + qz(t-\sigma) \\
&\quad = p\left[x(t-\tau) - p\int_{t-2\tau}^{t-\tau-\sigma} a(s)\,x(s)\,ds\right] \\
&\qquad + q\left[x(t-\sigma) - p\int_{t-\tau-\sigma}^{t-2\sigma} a(s)\,x(s)\,ds\right] \\
&\quad = px(t-\tau) + qx(t-\sigma) \\
&\qquad - p\left[p\int_{t-2\tau}^{t-\tau-\sigma} a(s)\,x(s)\,ds + q\int_{t-\tau-\sigma}^{t-2\sigma} a(s)\,x(s)\,ds\right] \\
&\quad = px(t-\tau) + qx(t-\sigma) \\
&\qquad - p\left[p\int_{t-\tau}^{t-\sigma} a(s-\tau)\,x(s-\tau)\,ds + q\int_{t-\tau}^{t-\sigma} a(s-\sigma)\,x(s-\sigma)\,ds\right] \\
&\quad = px(t-\tau) + qx(t-\sigma) \\
&\qquad - p\left[p\int_{t-\tau}^{t-\sigma} a(s)\,x(s-\tau)\,ds + q\int_{t-\tau}^{t-\sigma} a(s)\,x(s-\tau)\,ds\right] \\
&\quad = px(t-\tau) + qx(t-\sigma) \\
&\qquad - p\int_{t-\tau}^{t-\sigma} a(s)[px(s-\tau) + qx(s-\sigma)]\,ds \\
&\quad = px(t-\tau) + qx(t-\sigma) - p\int_{t-\tau}^{t-\sigma} [-x'(s)]\,ds \\
&\quad = px(t-\tau) + qx(t-\sigma) - p[x(t-\tau) - x(t-\sigma)]
\end{aligned}$$

and therefore

$$pz(t-\tau) + qz(t-\sigma) = (p+q)\,x(t-\sigma) \qquad \text{for all} \quad t \geqslant T_0. \tag{7}$$

Combining (6) and (7), we conclude that the function z is a solution on $[T_0, \infty)$ of the differential equation (E).

LEMMA 3. *Suppose that* (1), (2), *and* (3) *are satisfied. Let* x *be a positive solution on an interval* $[t_0, \infty)$, $t_0 \geqslant 0$, *of the differential equation* (E) *and* *define z as given by* (5). *Moreover, let T_0 be defined as in Lemma* 2. *Then the following statements are true*:

(i) *Assume that*

$$Ap(\tau-\sigma)\leqslant 1.$$

Then z is a solution of (E) *on the interval* $[T_0, \infty)$, *which is decreasing and positive.*

(ii) *Assume that*

$$Ap(\tau-\sigma)>1$$

and set $w=-z$. Then w is a solution of (E) *on the interval* $[T_0, \infty)$, *which is increasing and eventually positive.*

Note. A solution of (E) on an interval $[\xi, \infty)$, $\xi\geqslant 0$, is said to be positive or decreasing or increasing if it is positive or decreasing or increasing, respectively on the whole interval $[\xi-\rho, \infty)$, where $\rho=\max\{\tau, \sigma\}$. Also, a solution of (E) on an interval $[\xi, \infty)$, $\xi\geqslant 0$, is called eventually positive if it is positive on $[\hat{\xi}, \infty)$ for some $\hat{\xi}\geqslant\xi-\rho$.

Proof. First of all, we have

$$\int_{t-\tau}^{t} a(s)\,ds = A\tau \qquad \text{for} \quad t\geqslant\tau \tag{8}$$

and

$$\int_{t-\sigma}^{t} a(s)\,ds = A\sigma \qquad \text{for} \quad t\geqslant\sigma. \tag{9}$$

In fact, by taking into account the fact that the function a is ω-periodic and that $\tau=\mu\omega$, we obtain for $t\geqslant\tau$ (by (1), τ is positive)

$$\int_{t-\tau}^{t} a(s)\,ds=\int_{0}^{\tau} a(s)\,ds=\left[\frac{1}{\tau}\int_{0}^{\tau} a(s)\,ds\right]\tau=\left[\frac{1}{\omega}\int_{0}^{\omega} a(s)\,ds\right]\tau=A\tau.$$

By a parallel argument, we can establish (9).

(i) From Lemma 2 we know that z is a solution of (E) on the interval $[T_0, \infty)$. Moreover, from (2) and (6) we see that the function z is decreasing on $[\max\{t_0, \tau, \sigma\}, \infty)$. It remains to establish that z is positive on the interval $[\max\{t_0, \tau, \sigma\}, \infty)$. Since a is not identically zero on the interval $[0, \infty)$ and ω-periodic, it follows that the function a is not identically zero on any interval of the form $[\xi, \infty)$, $\xi \geqslant 0$. Thus, (2) and (6) imply that z is not eventually constant. Now, in order to prove that z is positive on the interval $[\max\{t_0, \tau, \sigma\}, \infty)$, it suffices to show that

$$\lim_{t \to \infty} z(t) = 0. \tag{10}$$

First, we claim that $\lim_{t \to \infty} z(t)$ is a finite number. Otherwise,

$$\lim_{t \to \infty} z(t) = -\infty, \tag{11}$$

which guarantees that z is eventually negative. Moreover, (11) implies that x is unbounded. Indeed, in the opposite case there exists a constant $K > 0$ such that $x(t) \leqslant K$ for all $t \geqslant t_0$-$\max\{\tau, \sigma\}$. Then, by taking into account (1), (3), (8), and (9), from (5) we obtain for $t \geqslant \max\{t_0, \tau, \sigma\}$

$$\begin{aligned} x(t) &= z(t) + p \int_{t-\tau}^{t-\sigma} a(s)\, x(s)\, ds \leqslant z(t) + pK \int_{t-\tau}^{t-\sigma} a(s)\, ds \\ &= z(t) + pK \left[\int_{t-\tau}^{t} a(s)\, ds - \int_{t-\sigma}^{t} a(s)\, ds \right] = z(t) + pKA(\tau - \sigma) \to -\infty \end{aligned}$$

as $t \to \infty$ which is a contradiction. Now, by taking into account the fact that z is eventually negative and x is unbounded, we can choose a point $t_1 \geqslant \max\{t_0, \tau, \sigma\}$ such that

$$z(t_1) < 0 \text{ and } x(t_1) = \max\{x(s) \colon t_0 - \max\{\tau, \sigma\} \leqslant s \leqslant t_1\}.$$

By using (1), (3), (8), and (9), from (5) we find

$$\begin{aligned} 0 > z(t_1) &= x(t_1) - p \int_{t_1-\tau}^{t_1-\sigma} a(s)\, x(s)\, ds \geqslant x(t_1) \left[1 - p \int_{t_1-\tau}^{t_1-\sigma} a(s)\, ds \right] \\ &= x(t_1) \left\{ 1 - p \left[\int_{t_1-\tau}^{t_1} a(s)\, ds - \int_{t_1-\sigma}^{t_1} a(s)\, ds \right] \right\} \\ &= x(t_1)[1 - Ap(\tau - \sigma)] \geqslant 0. \end{aligned}$$

This contradiction establishes our claim that $l \equiv \lim_{t \to \infty} z(t)$ is finite. Integrating both sides of (6) from $t_0^* \equiv \max\{t_0, \tau, \sigma\}$ to t and letting $t \to \infty$, we obtain

$$l - z(t_0^*) = -(p+q) \int_{t_0^*}^{\infty} a(s)\, x(s - \sigma)\, ds,$$

which gives

$$\int_{t_0^*}^{\infty} a(s)\, x(s-\sigma)\, ds < \infty. \tag{12}$$

But, by using the fact that the function a is ω-periodic and that $\tau = \mu\omega$ and $\sigma = \nu\omega$, we get

$$\begin{aligned}\int_{t_0^*}^{\infty} a(s)\, x(s-\sigma)\, ds &= \int_{t_0^*-\sigma}^{\infty} a(s+\sigma)\, x(s)\, ds = \int_{t_0^*-\sigma}^{\infty} a(s)\, x(s)\, ds \\ &= \int_{t_0^*-\sigma+\tau}^{\infty} a(s-\tau)\, x(s-\tau)\, ds = \int_{t_0^*-\sigma-\tau}^{\infty} a(s)\, x(s-\tau)\, ds.\end{aligned}$$

Hence, because of (12), we have

$$\int_{t_0^*}^{\infty} a(s)\, x(s)\, ds < \infty \tag{13}$$

and

$$\int_{t_0^*}^{\infty} a(s)\, x(s-\tau)\, ds < \infty. \tag{14}$$

In view of (12) and (14), from (E) it follows that $|\int_{t_0^*}^{\infty} x'(s)\, ds| < \infty$ and so $\lim_{t\to\infty} x(t)$ exists. We must have

$$\lim_{t\to\infty} x(t) = 0. \tag{15}$$

In fact, in the opposite case there exists a constant $c > 0$ so that $x(t) \geqslant c$ for $t \geqslant t_0^*$. Hence, (13) gives $\int_{t_0^*}^{\infty} a(s)\, ds < \infty$. This is a contradiction, since a is not identically zero on the interval $[0, \infty)$ and ω-periodic. From (13) we see

$$\lim_{t\to\infty} \int_{t-\tau}^{t-\sigma} a(s)\, x(s)\, ds = 0. \tag{16}$$

By (15) and (16), from (5) we conclude that (10) holds.

(ii) From Lemma 2 and the linearity of (E), it follows that w is a solution on $[T_0, \infty)$ of the differential equation (E). Also, from (6) we see that

$$w'(t) = a(t)(p+q)\, x(t-\sigma) \qquad \text{for} \quad t \geqslant \max\{t_0, \tau, \sigma\} \tag{17}$$

and so, in view of (2), we conclude that w is increasing on the interval

$[\max\{t_0, \tau, \sigma\}, \infty)$. To show that w is eventually positive it suffices to establish that

$$\lim_{t \to \infty} w(t) = \infty. \tag{18}$$

Otherwise, $l \equiv \lim_{t \to \infty} w(t)$ exists and is finite. By setting $t_0^* = \max\{t_0, \tau, \sigma\}$, from (17) we get

$$l - w(t_0^*) = (p+q) \int_{t_0^*}^{\infty} a(s)\, x(s-\sigma)\, ds,$$

which, by (2), guarantees that (12) is true. As in the proof of part (i), we see that (13) and (14) are satisfied and next we see that (15) and (16) hold. From (5), (15), and (16) it follows that $l \equiv \lim_{t \to \infty} w(t) = 0$. But, the function a is not identically zero on any interval of the form $[\xi, \infty)$, $\xi \geqslant 0$, and hence from (2) and (17) it follows that w is not eventually constant. Thus, w is negative on the interval $[\max\{t_0, \tau, \sigma\}, \infty)$ and so we can consider a point $t_1 \geqslant \max\{t_0, \tau, \sigma\}$ so that

$$w(t_1) < 0 \qquad \text{and} \qquad x(t_1) = \min\{x(s): t_0 - \max\{\tau, \sigma\} \leqslant s \leqslant t_1\}.$$

By using (1), (3), (8), and (9), from (5) we find

$$\begin{aligned} 0 > w(t_1) &= -x(t_1) + p \int_{t_1 - \tau}^{t_1 - \sigma} a(s)\, x(s)\, ds \geqslant x(t_1) \left[-1 + p \int_{t_1 - \tau}^{t_1 - \sigma} a(s)\, ds \right] \\ &= x(t_1) \left\{ -1 + p \left[\int_{t_1 - \tau}^{t_1} a(s)\, ds - \int_{t_1 - \sigma}^{t_1} a(s)\, ds \right] \right\} \\ &= x(t_1)[-1 + Ap(\tau - \sigma)] > 0. \end{aligned}$$

This contradiction shows that (18) holds.

3. Proof of Theorem 1

First of all, as in the proof of Lemma 3, we see that (8) holds when $\tau > 0$. On the other hand, (8) is obvious for $\tau = 0$. Hence, (8) is satisfied in both cases where $\tau > 0$ or $\tau = 0$. In a similar way, we find that (9) is true for $\sigma \geqslant 0$.

For the proof that (I) $\Rightarrow$ (II) assume, for the sake of contradiction, that $(*)$ has a real root λ_0. Define

$$x(t) = \exp\left[\frac{\lambda_0}{A} \int_0^t a(s)\, ds \right] \qquad \text{for} \quad t \geqslant 0.$$

Then, by using (8) and (9), we obtain for $t \geqslant \rho \equiv \max\{\tau, \sigma\}$

$$\begin{aligned} x'(t) &+ a(t)[px(t-\tau)+qx(t-\sigma)] \\ &= \frac{1}{A} a(t) \exp\left[\frac{\lambda_0}{A}\int_0^t a(s)\,ds\right] \\ &\quad \times \left\{\lambda_0 + A\left(p\exp\left[-\frac{\lambda_0}{A}\int_{t-\tau}^t a(s)\,ds\right] + q\exp\left[-\frac{\lambda_0}{A}\int_{t-\sigma}^t a(s)\,ds\right]\right)\right\} \\ &= \frac{1}{A} a(t)\exp\left[\frac{\lambda_0}{A}\int_0^t a(s)\,ds\right][\lambda_0 + A(pe^{-\lambda_0\tau} + qe^{-\lambda_0\sigma})] = 0 \end{aligned}$$

and hence x is a solution on $[\rho, \infty)$ of the differential equation (E). Clearly, this solution is positive. So, there is a nonoscillatory solution of (E). This contradicts the hypothesis that all solutions of (E) are oscillatory and completes the proof that (I) $\Rightarrow$ (II).

Next we will show that (II) $\Rightarrow$ (I). When $\sigma = \tau$, the differential equation (E) takes the form

$$x'(t) + a(t)(p+q)\,x(t-\tau) = 0$$

and (*) reduces to the equation

$$\lambda + A(p+q)\,e^{-\lambda\tau} = 0.$$

Also, if $\sigma = 0$, then the transformation

$$x(t) = y(t)\exp\left[-q\int_{-\rho}^t \tilde{a}(s)\,ds\right] \qquad \text{for} \quad t \geqslant -\rho$$

where $\tilde{a}$ is the ω-periodic extension of the function a on the interval $[-\rho, \infty)$, transforms (E) into the equation

$$y'(t) + a(t)\,p\exp\left[q\int_{t-\tau}^t \tilde{a}(s)\,ds\right]y(t-\tau) = 0.$$

By (8), the last equation becomes

$$y'(t) + a(t)\,pe^{qA\tau}y(t-\tau) = 0.$$

On the other hand, when $\sigma = 0$, the equation (*) becomes

$$(\lambda + Aq) + Ape^{-\lambda\tau} = 0.$$

By using the transformation $\lambda + Aq = \tilde{\lambda}$, the last equation can be written in the form

$$\tilde{\lambda} + (Ape^{qA\tau})\,e^{-\tilde{\lambda}\tau} = 0.$$

So, in both cases where $\sigma = \tau$ or $\sigma = 0$ the differential equation (E) reduces to an equation with one delay of the form

$$x'(t) + a(t)\, r x(t-\tau) = 0 \tag{E$_0$}$$

and the equation (*) becomes

$$\lambda + Are^{-\lambda\tau} = 0, \tag{$*$$_0$}$$

where r is a real number. Now, assume that $(*)_0$ has no real roots. Then it is obvious that $\tau > 0$. Moreover, as $\lim_{\lambda \to \infty} (\lambda + Are^{-\lambda\tau}) = \infty$, it follows that

$$F_0(\lambda) \equiv \lambda + Are^{-\lambda\tau} > 0 \qquad \text{for all} \quad \lambda \in \mathbb{R}.$$

In particular, we have $F_0(0) > 0$ and hence $r > 0$. Next, we find

$$\min_{\lambda \in \mathbb{R}} F_0(\lambda) = \frac{1}{\tau} \ln(Ar\tau e) > 0$$

and consequently

$$Ar\tau > 1/e.$$

Thus, in view of (8), we have

$$\liminf_{t \to \infty} \int_{t-\tau}^{t} a(s)\, r\, ds > \frac{1}{e}.$$

But it is known (see [5, 6]) that the last inequality is a sufficient condition for the oscillation of all solutions of $(\mathrm{E})_0$.

Hence, in what follows we may (and do) assume that the delays τ and σ satisfy (1).

Suppose that (*) has no real roots. Then, since (1) is assumed to hold, Lemma 1 implies that (2), (3), and (4) are true. Furthermore, assume for the sake of contradiction that the differential equation (E) has a nonoscillatory solution x on an interval $[t_0, \infty)$, $t_0 \geqslant 0$. Without loss of generality, we may assume that $x(t) \neq 0$ for all $t \geqslant t_0 - \rho$, where $\rho = \max\{\tau, \sigma\}$. As the negative of a solution of (E) is also a solution of the same equation, we will confine our discussion to the case where x is positive on $[t_0 - \rho, \infty)$.

In the sequel, for convenience, we will suppose that inequalities about values of functions are satisfied eventually for all large t.

We distinguish the following two cases:

Case 1. $Ap(\tau-\sigma)\leqslant 1$. Setting

$$z(t)=x(t)-p\int_{t-\tau}^{t-\sigma} a(s)\,x(s)\,ds,$$

we know, by Lemma 3(i), that z is a solution of (E) which is eventually positive and decreasing. Furthermore, from (6), we have

$$z'(t)+a(t)(p+q)\,x(t-\sigma)=0.$$

Define

$$z_0(t)=z(t)$$

and

$$z_n(t)=z_{n-1}(t)-p\int_{t-\tau}^{t-\sigma} a(s)\,z_{n-1}(s)\,ds \qquad (n=1,2,...). \tag{19}$$

Then, for every $n\in\{1,2,...\}$, the function z_n is a solution of (E) which is eventually positive and decreasing. Moreover, one has

$$z_n'(t)+a(t)(p+q)\,z_{n-1}(t-\sigma)=0 \qquad (n=1,2,...). \tag{20}$$

Next, for each $n=1,2,...,$ we introduce the set

$$\Lambda(z_n)=\left\{\lambda>0\colon z_n'(t)+\frac{\lambda}{A}\,a(t)\,z_n(t)\leqslant 0\right\}.$$

The proof will be accomplished by proving that $\Lambda(z_n)$ has the following contradictory properties:

(P_1) For each $n=1,2,...,$ the set $\Lambda(z_n)$ is nonempty and bounded above by a number independent of n.

(P_2) If $n\in\{1,2,...\}$, then $\lambda\in\Lambda(z_n)\Rightarrow(\lambda+m)\in\Lambda(z_{n+1})$, where m is the positive number defined by (4).

Consider an arbitrary positive integer n. It is clear that

$$z_n(t)\leqslant z_{n-1}(t)\leqslant z_{n-1}(t-\sigma)$$

and so (20) yields

$$z_n'(t)+(p+q)\,a(t)\,z_n(t)\leqslant 0.$$

Thus, we see that $A(p+q)\in\Lambda(z_n)$, which shows that $\Lambda(z_n)\neq\varnothing$.

Next, we consider an arbitrary $n \in \{1, 2, ...\}$ and we prove that $A(z_n)$ is bounded above by a number which is independent of n. Indeed, integrating both sides of (20) from $t-\sigma$ to t, we get

$$z_n(t) - z_n(t-\sigma) + (p+q) \int_{t-\sigma}^{t} a(s)\, z_{n-1}(s-\sigma)\, ds = 0,$$

which gives

$$-z_n(t-\sigma) + (p+q) \left[\int_{t-\sigma}^{t} a(s)\, ds \right] z_{n-1}(t-\sigma) < 0$$

and therefore, in view of (9), we obtain

$$-z_n(t-\sigma) + (p+q)\, A\sigma z_{n-1}(t-\sigma) < 0.$$

So,

$$z_{n-1}(t) < \frac{1}{(p+q)\, A\sigma}\, z_n(t). \tag{21}$$

By using (8) and (9), from (19) we find that

$$\begin{aligned} z_n(t) &\leqslant z_{n-1}(t) - p \left[\int_{t-\tau}^{t-\sigma} a(s)\, ds \right] z_{n-1}(t-\sigma) \\ &= z_{n-1}(t) - p \left[\int_{t-\tau}^{t} a(s)\, ds - \int_{t-\sigma}^{t} a(s)\, ds \right] z_{n-1}(t-\sigma) \\ &= z_{n-1}(t) - pA(\tau-\sigma)\, z_{n-1}(t-\sigma) \end{aligned}$$

and hence, by (21), we obtain

$$pA(\tau-\sigma)\, z_{n-1}(t-\sigma) \leqslant z_{n-1}(t) - z_n(t) < z_{n-1}(t) < \frac{1}{(p+q)\, A\sigma}\, z_n(t).$$

Therefore,

$$(p+q)\, z_{n-1}(t-\sigma) < \frac{1}{pA^2(\tau-\sigma)\sigma}\, z_n(t)$$

and so (20) implies that

$$z_n'(t) + \frac{1}{pA^2(\tau-\sigma)\sigma}\, a(t)\, z_n(t) > 0.$$

This proves that the number $1/[pA(\tau-\sigma)\sigma]$ is not an element of the set $A(z_n)$. That is, $A(z_n)$ is bounded above by $1/[pA(\tau-\sigma)\sigma]$.

Now, we will establish (P_2). Let $n \in \{1, 2, ...\}$ and let λ be an arbitrary number in $\Lambda(z_n)$. Set

$$\varphi_n(t) = z_n(t) \exp\left[\frac{\lambda}{A}\int_0^t a(s)\,ds\right].$$

Then

$$\varphi_n'(t) = \left[z_n'(t) + \frac{\lambda}{A} a(t)\, z_n(t)\right] \exp\left[\frac{\lambda}{A}\int_0^t a(s)\,ds\right] \leqslant 0$$

and from (19) we obtain

$$\begin{aligned}
z_{n+1}(t) &= z_n(t) - p\int_{t-\tau}^{t-\sigma} a(s)\, z_n(s)\,ds \\
&= \varphi_n(t)\exp\left[-\frac{\lambda}{A}\int_0^t a(s)\,ds\right] \\
&\quad - p\int_{t-\tau}^{t-\sigma} a(s)\,\varphi_n(s)\exp\left[-\frac{\lambda}{A}\int_0^s a(u)\,du\right] ds \\
&\leqslant \varphi_n(t-\sigma)\exp\left[-\frac{\lambda}{A}\int_0^t a(s)\,ds\right] \\
&\quad - p\left\{\int_{t-\tau}^{t-\sigma} a(s)\exp\left[-\frac{\lambda}{A}\int_0^s a(u)\,du\right] ds\right\}\varphi_n(t-\sigma) \\
&= \varphi_n(t-\sigma)\exp\left[-\frac{\lambda}{A}\int_0^t a(s)\,ds\right] \\
&\quad - \frac{Ap}{\lambda}\left\{\exp\left[-\frac{\lambda}{A}\int_0^{t-\tau} a(s)\,ds\right]\right. \\
&\quad \left. - \exp\left[-\frac{\lambda}{A}\int_0^{t-\sigma} a(s)\,ds\right]\right\}\varphi_n(t-\sigma) \\
&= \left(1 - \frac{Ap}{\lambda}\left\{\exp\left[\frac{\lambda}{A}\int_{t-\tau}^t a(s)\,ds\right] - \exp\left[\frac{\lambda}{A}\int_{t-\sigma}^t a(s)\,ds\right]\right\}\right) \\
&\quad \times \varphi_n(t-\sigma)\exp\left[-\frac{\lambda}{A}\int_0^t a(s)\,ds\right].
\end{aligned}$$

So, in view of (8) and (9), we have

$$z_{n+1}(t) \leqslant \left[1 - \frac{Ap}{\lambda}(e^{\lambda\tau} - e^{\lambda\sigma})\right]\varphi_n(t-\sigma)\exp\left[-\frac{\lambda}{A}\int_0^t a(s)\,ds\right]. \tag{22}$$

Also, from (20) we find

$$\begin{aligned} z'_{n+1}(t) &= -a(t)(p+q)\, z_n(t-\sigma) \\ &= -a(t)(p+q)\, \varphi_n(t-\sigma) \exp\left[-\frac{\lambda}{A}\int_0^{t-\sigma} a(s)\, ds\right] \\ &= -a(t)(p+q)\, \varphi_n(t-\sigma) \exp\left[\frac{\lambda}{A}\int_{t-\sigma}^{t} a(s)\, ds\right] \\ &\quad \times \exp\left[-\frac{\lambda}{A}\int_0^{t} a(s)\, ds\right] \end{aligned}$$

and hence, by (9), we get

$$z'_{n+1}(t) = -a(t)(p+q)\, e^{\lambda\sigma}\varphi_n(t-\sigma) \exp\left[-\frac{\lambda}{A}\int_0^{t} a(s)\, ds\right]. \tag{23}$$

So, from (22) and (23) we obtain

$$\begin{aligned} & z'_{n+1}(t) + \frac{\lambda+m}{A}\, a(t)\, z_{n+1}(t) \\ &\leqslant a(t)\left\{-(p+q)\, e^{\lambda\sigma} + \frac{\lambda+m}{A}\left[1 - \frac{Ap}{\lambda}(e^{\lambda\tau} - e^{\lambda\sigma})\right]\right\} \\ &\quad \times \varphi_n(t-\sigma) \exp\left[-\frac{\lambda}{A}\int_0^{t} a(s)\, ds\right] \\ &= \frac{1}{A}\, a(t)\left\{m - [(-\lambda) + A(pe^{\lambda\tau} + qe^{\lambda\sigma})] - \frac{mpA}{\lambda}(e^{\lambda\tau} - e^{\lambda\sigma})\right\} \\ &\quad \times \varphi_n(t-\sigma) \exp\left[-\frac{\lambda}{A}\int_0^{t} a(s)\, ds\right] \\ &\leqslant 0, \end{aligned}$$

which proves that $(\lambda+m) \in \Lambda(z_{n+1})$ and completes the proof in this case.

Case 2. $Ap(\tau-\sigma) > 1$. If we define

$$w(t) = -z(t) = -x(t) + p\int_{t-\tau}^{t-\sigma} a(s)\, x(s)\, ds,$$

then Lemma 3(ii) ensures that w is a solution of (E) which is eventually positive and increasing. Also, (17) gives

$$w'(t) - a(t)(p+q)\, x(t-\sigma) = 0.$$

Set

$$w_0(t) = w(t)$$

and

$$w_n(t) = -w_{n-1}(t) + p \int_{t-\tau}^{t-\sigma} a(s)\, w_{n-1}(s)\, ds \qquad (n = 1, 2, ...). \tag{24}$$

Then, for each $n = 1, 2, ...,$ the function w_n is also a solution of (E) which is eventually positive and increasing. Moreover, one has

$$w_n'(t) - a(t)(p+q)\, w_{n-1}(t-\sigma) = 0 \qquad (n = 1, 2, ...). \tag{25}$$

For any $n = 1, 2, ...,$ we define the set

$$\Lambda(w_n) = \left\{ \lambda > 0 \colon w_n'(t) - \frac{\lambda}{A}\, a(t)\, w_n(t) \geqslant 0 \right\}.$$

Also, we set

$$\lambda_0 = \frac{p+q}{p(\tau - \sigma)} \qquad \text{and} \qquad m_0 = \frac{m\lambda_0}{Ap}\, e^{\lambda_0 \sigma}.$$

The proof will be accomplished by proving that $\Lambda(w_n)$ admits the following contradictory properties:

(P_1) For each $n = 1, 2, ...,$ the set $\Lambda(w_n)$ is nonempty and bounded above by a number independent of n.

(P_2) For any $n = 1, 2, ...$ and every $\lambda \in \Lambda(w_n)$ with $\lambda \geqslant \lambda_0$, the number $\lambda + m_0$ belongs also in $\Lambda(w_{n+1})$.

First we will show (P_1). Let $n \in \{1, 2, ...\}$. By using (8), (9), and (24), we get

$$\begin{aligned} w_n(t) &\leqslant p \int_{t-\tau}^{t-\sigma} a(s)\, w_{n-1}(s)\, ds \leqslant p \left[\int_{t-\tau}^{t-\sigma} a(s)\, ds \right] w_{n-1}(t-\sigma) \\ &= p \left[\int_{t-\tau}^{t} a(s)\, ds - \int_{t-\sigma}^{t} a(s)\, ds \right] w_{n-1}(t-\sigma) = pA(\tau - \sigma)\, w_{n-1}(t-\sigma) \end{aligned}$$

and hence, by (25), we find that

$$w_n'(t) - a(t)(p+q) \frac{1}{pA(\tau - \sigma)}\, w_n(t) \geqslant 0.$$

This means that $\lambda_0 \in \Lambda(w_n)$ and so $\Lambda(w_n) \neq \varnothing$. Now, from (25) and the fact that w_n is a solution of (E), we obtain

$$a(t)(p+q)\, w_{n-1}(t-\sigma) = w_n'(t) = -a(t)[\, p w_n(t-\tau) + q w_n(t-\sigma)]. \tag{26}$$

But the function a is not identically zero on any interval of the form $[\xi, \infty)$, $\xi \geqslant 0$. So, there exists a sequence $(t_r)_{r=1,2,\dots}$ with $\lim_{r\to\infty} t_r = \infty$ and such that

$$a(t_r) > 0 \qquad \text{for} \quad r = 1, 2, \dots .$$

Then (26) implies that

$$(p+q)\, w_{n-1}(t_r - \sigma) = -[p w_n(t_r - \tau) + q w_n(t_r - \sigma)]$$

for $r = 1, 2, \dots$, which means that $q < 0$ and

$$(p+q)\, w_{n-1}(t_r - \sigma) < -q w_n(t_r - \sigma) < -q w_n(t_r).$$

Using this in (25), we get

$$w_n'(t_r) - (-q)\, a(t_r)\, w_n(t_r) < 0 \qquad (r = 1, 2, \dots),$$

which proves that $(-qA) \notin \Lambda(w_n)$. That is, the number $-qA$ is an upper bound of $\Lambda(w_n)$ which is independent of n.

Finally, we will establish (P_2). Let n be an arbitrary positive integer and let λ be an arbitrary element of $\Lambda(w_n)$ with $\lambda \geqslant \lambda_0$. Set

$$\psi_n(t) = w_n(t) \exp\left[-\frac{\lambda}{A}\int_0^t a(s)\, ds\right].$$

Then

$$\psi_n'(t) = \left[w_n'(t) - \frac{\lambda}{A} a(t)\, w_n(t)\right] \exp\left[-\frac{\lambda}{A}\int_0^t a(s)\, ds\right] \geqslant 0.$$

Thus, from (24) we obtain

$$\begin{aligned}
w_{n+1}(t) &= -w_n(t) + p\int_{t-\tau}^{t-\sigma} a(s)\, w_n(s)\, ds \\
&= -\psi_n(t) \exp\left[\frac{\lambda}{A}\int_0^t a(s)\, ds\right] \\
&\quad + p\int_{t-\tau}^{t-\sigma} a(s)\, \psi_n(s) \exp\left[\frac{\lambda}{A}\int_0^s a(u)\, du\right] ds \\
&\leqslant -\psi_n(t-\sigma) \exp\left[\frac{\lambda}{A}\int_0^t a(s)\, ds\right] \\
&\quad + p\left\{\int_{t-\tau}^{t-\sigma} a(s) \exp\left[\frac{\lambda}{A}\int_0^s a(u)\, du\right] ds\right\} \psi_n(t-\sigma)
\end{aligned}$$

$$= -\psi_n(t-\sigma)\exp\left[\frac{\lambda}{A}\int_0^t a(s)\,ds\right]$$

$$+\frac{Ap}{\lambda}\left\{\exp\left[\frac{\lambda}{A}\int_0^{t-\sigma} a(s)\,ds\right]-\exp\left[\frac{\lambda}{A}\int_0^{t-\tau} a(s)\,ds\right]\right\}\psi_n(t-\sigma)$$

$$=\left(-1+\frac{Ap}{\lambda}\left\{\exp\left[-\frac{\lambda}{A}\int_{t-\sigma}^{t} a(s)\,ds\right]\right.\right.$$

$$\left.\left.-\exp\left[-\frac{\lambda}{A}\int_{t-\tau}^{t} a(s)\,ds\right]\right\}\right)\psi_n(t-\sigma)\cdot\exp\left[\frac{\lambda}{A}\int_0^t a(s)\,ds\right]$$

and so, by using (8) and (9), we get

$$w_{n+1}(t)\leqslant\left[-1+\frac{Ap}{\lambda}(e^{-\lambda\sigma}-e^{-\lambda\tau})\right]\psi_n(t-\sigma)\exp\left[\frac{\lambda}{A}\int_0^t a(s)\,ds\right]. \qquad (27)$$

Furthermore, (25) yields

$$w'_{n+1}(t)=a(t)(p+q)\,w_n(t-\sigma)$$

$$=a(t)(p+q)\,\psi_n(t-\sigma)\exp\left[\frac{\lambda}{A}\int_0^{t-\sigma} a(s)\,ds\right]$$

$$=a(t)(p+q)\,\psi_n(t-\sigma)\exp\left[-\frac{\lambda}{A}\int_{t-\sigma}^{t} a(s)\,ds\right]$$

$$\times\exp\left[\frac{\lambda}{A}\int_0^t a(s)\,ds\right]$$

and consequently, in view of (9), we have

$$w'_{n+1}(t)=a(t)(p+q)\,\mathrm{e}^{-\lambda\sigma}\psi_n(t-\sigma)\exp\left[\frac{\lambda}{A}\int_0^t a(s)\,ds\right]. \qquad (28)$$

Combining (27) and (28), we find

$$w'_{n+1}(t)-\frac{\lambda+m_0}{A}a(t)\,w_{n+1}(t)$$

$$\geqslant a(t)\left\{(p+q)\,e^{-\lambda\sigma}-\frac{\lambda+m_0}{A}\left[-1+\frac{Ap}{\lambda}(e^{-\lambda\sigma}-e^{-\lambda\tau})\right]\right\}$$

$$\times\psi_n(t-\sigma)\exp\left[\frac{\lambda}{A}\int_0^t a(s)\,ds\right]$$

$$= \frac{1}{A} a(t) \left\{ \lambda + A(pe^{-\lambda\tau} + qe^{-\lambda\sigma}) + m_0 - \frac{m_0 Ap}{\lambda}(e^{-\lambda\sigma} - e^{-\lambda\tau}) \right\}$$

$$\times \psi_n(t-\sigma) \exp\left[\frac{\lambda}{A}\int_0^t a(s)\, ds\right]$$

$$\geqslant \frac{1}{A} a(t) \left\{ [\lambda + A(pe^{-\lambda\tau} + qe^{-\lambda\sigma})] - \frac{m_0 Ap}{\lambda} e^{-\lambda\sigma} \right\}$$

$$\times \psi_n(t-\sigma) \exp\left[\frac{\lambda}{A}\int_0^t a(s)\, ds\right]$$

$$\geqslant \frac{1}{A} a(t) \left\{ [\lambda + A(pe^{-\lambda\tau} + qe^{-\lambda\sigma})] - \frac{m_0 Ap}{\lambda_0} e^{-\lambda_0\sigma} \right\}$$

$$\times \psi_n(t-\sigma) \exp\left[\frac{\lambda}{A}\int_0^t a(s)\, ds\right]$$

$$\geqslant \frac{1}{A} a(t) \left\{ [\lambda + A(pe^{-\lambda\tau} + qe^{-\lambda\sigma})] - m \right\}$$

$$\times \psi_n(t-\sigma) \exp\left[\frac{\lambda}{A}\int_0^t a(s)\, ds\right] \geqslant 0.$$

This implies that $(\lambda + m_0) \in \Lambda(w_{n+1})$.

The proof of the theorem is now complete.

4. Proofs of Theorems 2 and 3

In this section we prove Theorems 2 and 3. Note that in the proof of Theorem 3 we use Theorem 2.

Proof of Theorem 2. We have $f - g \geqslant (c-1)g$ and hence the function $f - g$ is nonnegative on the interval $[0, \infty)$ and not identically zero on this interval. Therefore, we can see that

$$F > G. \tag{29}$$

Furthermore, from condition (C_1) it follows that

$$\tau \geqslant \sigma. \tag{30}$$

Let x be a nonoscillatory solution on an interval $[t_0, \infty)$, $t_0 \geqslant 0$, of the differential equation $(\tilde{E})$. Without loss of generality, we can suppose that $x(t) \neq 0$ for all $t \geqslant t_0 - \tau$. Furthermore, as the negative of a solution of $(\tilde{E})$

is also a solution of the same equation, we may (and do) assume that x is positive on the interval $[t_0-\tau, \infty)$. Set $T_0=\max\{t_0, \tau\}$ and define

$$z(t)=x(t)-\int_{t-\tau}^{t-\sigma} g(s)\, x(s)\, ds \qquad \text{for} \quad t \geqslant T_0. \tag{31}$$

Then, by taking into account the fact that the function g is ω-periodic and that $\tau=\mu\omega$ and $\sigma=v\omega$, from (31) and ($\tilde{\mathrm{E}}$) we obtain for $t \geqslant T_0$

$$\begin{aligned} z'(t) &= x'(t)-[g(t-\sigma)\, x(t-\sigma)-g(t-\tau)\, x(t-\tau)] \\ &= -[f(t)\, x(t-\tau)-g(t)\, x(t-\sigma)]-[g(t)\, x(t-\sigma)-g(t)\, x(t-\tau)] \end{aligned}$$

and consequently

$$z'(t)= -[f(t)-g(t)]\, x(t-\tau) \qquad \text{for every} \quad t \geqslant T_0. \tag{32}$$

Since $f-g$ is nonnegative on $[0, \infty)$, it follows from (32) that the function z is decreasing on the interval $[T_0, \infty)$. We claim that z is bounded below. In the opposite case, we have

$$\lim_{t \to \infty} z(t)= -\infty \tag{33}$$

and hence z is eventually negative. On the other hand, the function x is unbounded. Indeed, otherwise there exists a positive constant K with $x(t) \leqslant K$ for $t \geqslant t_0-\tau$. Thus, by using (30), from (31) we find

$$x(t) \leqslant z(t)+K\int_{t-\tau}^{t-\sigma} g(s)\, ds \qquad \text{for every} \quad t \geqslant T_0. \tag{34}$$

But we have

$$\int_{t-\tau}^{t} g(s)\, ds=G\tau \qquad \text{for} \quad t \geqslant \tau \tag{35}$$

and

$$\int_{t-\sigma}^{t} g(s)\, ds=G\sigma \qquad \text{for} \quad t \geqslant \sigma. \tag{36}$$

In fact, (35) is obvious when $\tau=0$. So we suppose that $\tau>0$. Then, by taking into account the fact that the function g is ω-periodic and that $\tau=\mu\omega$, we obtain for $t \geqslant \tau$

$$\int_{t-\tau}^{t} g(s)\, ds=\int_0^{\tau} g(s)\, ds=\left[\frac{1}{\tau}\int_0^{\tau} g(s)\, ds\right]\tau=\left[\frac{1}{\omega}\int_0^{\omega} g(s)\, ds\right]\tau=G\tau.$$

By a parallel argument we can establish (36). By (35) and (36), from (34) we get for $t \geqslant T_0$

$$x(t) \leqslant z(t) + K\left[\int_{t-\tau}^{t} g(s)\,ds - \int_{t-\sigma}^{t} g(s)\,ds\right] = z(t) + KG(\tau - \sigma)$$

and so, in view of (33), we arrive at the contradiction $\lim_{t\to\infty} x(t) = -\infty$. This contradiction establishes that x is unbounded. Now, since z is eventually negative and x is unbounded, we can consider a point $t_1 \geqslant T_0$ so that

$$z(t_1) < 0 \qquad \text{and} \qquad x(t_1) = \max\{x(s)\colon t_0 - \tau \leqslant s \leqslant t_1\}.$$

In view of (30), (35), (36), and condition (C_1), from (31) we obtain

$$\begin{aligned} 0 > z(t_1) &= x(t_1) - \int_{t_1-\tau}^{t_1-\sigma} g(s)\,x(s)\,ds \geqslant x(t_1) - x(t_1)\int_{t_1-\tau}^{t_1-\sigma} g(s)\,ds \\ &= x(t_1) - x(t_1)\left[\int_{t_1-\tau}^{t_1} g(s)\,ds - \int_{t_1-\sigma}^{t_1} g(s)\,ds\right] \\ &= x(t_1)[1 - G(\tau - \sigma)] \geqslant 0. \end{aligned}$$

This contradiction shows that z is bounded below and so $l \equiv \lim_{t\to\infty} z(t)$ exists and is finite. Now, integrating both sides of (32) from T_0 to t and letting $t \to \infty$, we derive

$$l - z(T_0) = -\int_{T_0}^{\infty} [f(s) - g(s)]\,x(s-\tau)\,ds,$$

which implies that

$$\int_{T_0}^{\infty} [f(s) - g(s)]\,x(s-\tau)\,ds < \infty. \tag{37}$$

But we have

$$\int_{T_0}^{\infty} [f(s) - g(s)]\,x(s-\tau)\,ds \geqslant (c-1)\int_{T_0}^{\infty} g(s)\,x(s-\tau)\,ds$$

with $c > 1$. Hence, (37) gives

$$\int_{T_0}^{\infty} g(s)\,x(s-\tau)\,ds < \infty. \tag{38}$$

From (37) and (38), it follows that

$$\int_{T_0}^{\infty} f(s)\,x(s-\tau)\,ds < \infty. \tag{39}$$

By using the fact that g is ω-periodic and that $\tau = \mu\omega$ and $\sigma = \nu\omega$, we derive

$$\int_{T_0}^{\infty} g(s)\, x(s-\tau)\, ds = \int_{T_0-\tau+\sigma}^{\infty} g(s-\sigma+\tau)\, x(s-\sigma)\, ds$$
$$= \int_{T_0-\tau+\sigma}^{\infty} g(s)\, x(s-\sigma)\, ds$$

and hence (38) implies that

$$\int_{T_0}^{\infty} g(s)\, x(s-\sigma)\, ds < \infty. \tag{40}$$

By using (39) and (40), from $(\tilde{\mathrm{E}})$ we obtain $|\int_{T_0}^{\infty} x'(s)\, ds| < \infty$ and so $\lim_{t\to\infty} x(t)$ exists. If $\lim_{t\to\infty} x(t)$ is positive, then there exists a constant $M > 0$ with $x(t) \geqslant M$ for all $t \geqslant T_0 - \tau$ and hence (39) implies that

$$\int_{T_0}^{\infty} f(s)\, ds < \infty,$$

which is a contradiction since the function f is ω-periodic and not identically zero on $[0, \infty)$. This contradiction shows that $\lim_{t\to\infty} x(t)$ must be zero and the proof is complete.

Proof of Theorem 3. From the inequality $f - g \geqslant (c-1)g$ it follows that the function $f-g$ is nonnegative on the interval $[0, \infty)$ and not identically zero on this interval. Since $f-g$ is ω-periodic, we can see that $f-g$ is not identically zero on any interval of the form $[\xi, \infty)$, $\xi \geqslant 0$.

Let x be a nonoscillatory solution of (E) on an interval $[t_0, \infty)$, $t_0 \geqslant 0$. As in the proof of Theorem 2, we can assume that x is positive on the interval $[t_0 - \tau, \infty)$ (note that condition (C_1) implies (30), i.e., $\tau \geqslant \sigma$). We consider the function z defined by (31), where $T_0 = \max\{t_0, \tau\}$. Then (32) holds and hence z is decreasing on $[T_0, \infty)$ and not eventually constant. Now, Theorem 2 ensures that

$$\lim_{t\to\infty} x(t) = 0. \tag{41}$$

On the other hand, by taking into account (30), from (31) we obtain for every $t \geqslant T_0$

$$z(t) \geqslant x(t) - [\max_{t-\tau \leqslant s \leqslant t-\sigma} x(s)] \int_{t-\tau}^{t-\sigma} g(s)\, ds$$
$$= x(t) - [\max_{t-\tau \leqslant s \leqslant t-\sigma} x(s)] \left[\int_{t-\tau}^{t} g(s)\, ds - \int_{t-\sigma}^{t} g(s)\, ds\right]$$

and so, by using (35) and (36), we get

$$z(t) \geqslant x(t) - G(\tau - \sigma)\left[\max_{t-\tau \leqslant s \leqslant t-\sigma} x(s)\right] \qquad \text{for all} \quad t \geqslant T_0. \tag{42}$$

In a similar manner, we can derive

$$z(t) \leqslant x(t) - G(\tau - \sigma)\left[\min_{t-\tau \leqslant s \leqslant t-\sigma} x(s)\right] \qquad \text{for all} \quad t \geqslant T_0. \tag{43}$$

From (41), (42) and (43) we find

$$\lim_{t \to \infty} z(t) = 0. \tag{44}$$

As z is decreasing on $[T_0, \infty)$ and not eventually constant, (44) implies that z is positive on the interval $[T_0, \infty)$. Furthermore, from (31) it follows that $z \leqslant x$ on $[T_0, \infty)$. Hence, (32) gives

$$z'(t) + [f(t) - g(t)]\, z(t - \tau) \leqslant 0 \tag{45}$$

for all $t \geqslant T_0 + \tau$. But, (35) holds and in a similar way we can prove that

$$\int_{t-\tau}^{t} f(s)\, ds = F\tau \qquad \text{for} \quad t \geqslant \tau. \tag{46}$$

Thus, in view of (35), (46), and condition (C_2), we have

$$\liminf_{t \to \infty} \int_{t-\tau}^{t} [f(s) - g(s)]\, ds = \tau(F - G) > \frac{1}{e}.$$

But, the last condition guarantees (see [5, 6, 9]) that the differential inequality (45) has no eventually positive solutions. So the proof of the theorem is complete.

5. A Remark

As we have proved in the proof of Theorem 2, the assumption $f \geqslant cg$ for some constant $c > 1$ implies that (29) holds, i.e., $F > G$. We observe that (29) is also a consequence of condition (C_2). Moreover, condition (C_1) implies (30), i.e., $\tau \geqslant \sigma$.

It is remarkable that (29) *and* (30) *are necessary conditions for all solutions of* $(\tilde{\mathrm{E}})$ *to be oscillatory*. To prove this result, we first claim that *a necessary condition for the oscillation of all solutions of* $(\tilde{\mathrm{E}})$ *is that the equation*

$$\lambda + Fe^{-\lambda\tau} - Ge^{-\lambda\sigma} = 0 \tag{47}$$

has no real roots. Assume that (47) has a real root λ and define

$$h_\lambda(t) = f(t)\, e^{-\lambda\tau} - g(t)\, e^{-\lambda\sigma} \qquad \text{for} \quad t \geqslant 0.$$

By using (35) and (46), we obtain for every $t \geqslant \tau$

$$\begin{aligned}\int_{t-\tau}^{t} h_\lambda(s)\, ds &= \left[\int_{t-\tau}^{t} f(s)\, ds\right] e^{-\lambda\tau} - \left[\int_{t-\tau}^{t} g(s)\, ds\right] e^{-\lambda\sigma}\\ &= F\tau e^{-\lambda\tau} - G\tau e^{-\lambda\sigma}\end{aligned}$$

and consequently

$$\int_{t-\tau}^{t} h_\lambda(s)\, ds = -\lambda\tau \qquad \text{for} \quad t \geqslant \tau. \tag{48}$$

In a similar manner, we can derive

$$\int_{t-\sigma}^{t} h_\lambda(s)\, ds = -\lambda\sigma \qquad \text{for} \quad t \geqslant \sigma. \tag{49}$$

Next, we set

$$x(t) = \exp\left[-\int_0^t h_\lambda(s)\, ds\right] \qquad \text{for} \quad t \geqslant 0.$$

Then, by (48) and (49), we obtain for $t \geqslant \rho \equiv \max\{\tau, \sigma\}$

$$\begin{aligned}&x'(t) + f(t)\, x(t-\tau) - g(t)\, x(t-\sigma)\\ &\quad = -h_\lambda(t) \exp\left[-\int_0^t h_\lambda(s)\, ds\right] + f(t) \exp\left[-\int_0^{t-\tau} h_\lambda(s)\, ds\right]\\ &\qquad - g(t) \exp\left[-\int_0^{t-\sigma} h_\lambda(s)\, ds\right]\\ &\quad = \left\{-h_\lambda(t) + f(t) \exp\left[\int_{t-\tau}^{t} h_\lambda(s)\, ds\right]\right.\\ &\qquad \left.- g(t) \exp\left[\int_{t-\sigma}^{t} h_\lambda(s)\, ds\right]\right\} \exp\left[-\int_0^t h_\lambda(s)\, ds\right]\\ &\quad = [-h_\lambda(t) + f(t)\, e^{-\lambda\tau} - g(t)\, e^{-\lambda\sigma}] \exp\left[-\int_0^t h_\lambda(s)\, ds\right] = 0\end{aligned}$$

and hence x is a solution on $[\rho, \infty)$ of the differential equation $(\tilde{\mathrm{E}})$, which is positive on $[0, \infty)$. So, there exists a nonoscillatory solution of $(\tilde{\mathrm{E}})$ and hence our claim is true. Now, suppose that all solutions of $(\breve{\mathrm{E}})$ are oscillatory. By our claim, (47) has no real roots. Set

$$H(\lambda) = \lambda + Fe^{-\lambda\tau} - Ge^{-\lambda\sigma} \qquad \text{for} \quad \lambda \in \mathbb{R}.$$

Since $H(\infty) = \infty$, it follows that

$$H(\lambda) > 0 \qquad \text{for all} \quad \lambda \in \mathbb{R}. \tag{50}$$

In particular, we have $H(0) = F - G > 0$ from which (29) follows. If $\tau < \sigma$, then we can see that $H(-\infty) = -\infty$, which contradicts (50). This contradiction establishes (30) and completes the proof.

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